

# **Nonlinear Partial Differential Equations with Applications**

Tomáš Roubíček

# Contents

<b>Preface</b>	<b>xi</b>
<b>Notational conventions</b>	<b>xv</b>
<b>1 Preliminary general material</b>	<b>1</b>
1.1 Functional analysis . . . . .	1
1.1.1 Normed spaces, Banach spaces, locally convex spaces . . . .	1
1.1.2 Functions and mappings on Banach spaces, dual spaces . .	3
1.1.3 Convex sets . . . . .	6
1.1.4 Compactness . . . . .	7
1.1.5 Fixed-point theorems . . . . .	8
1.2 Function spaces . . . . .	8
1.2.1 Continuous and smooth functions . . . . .	9
1.2.2 Lebesgue integrable functions . . . . .	10
1.2.3 Sobolev spaces . . . . .	14
1.3 Nemytskiĭ mappings . . . . .	19
1.4 Green formula and some inequalities . . . . .	20
1.5 Bochner spaces . . . . .	22
1.6 Some ordinary differential equations . . . . .	25
<b>I STEADY-STATE PROBLEMS</b>	<b>27</b>
<b>2 Pseudomonotone or weakly continuous mappings</b>	<b>29</b>
2.1 Abstract theory, basic definitions, Galerkin method . . . . .	29
2.2 Some facts about pseudomonotone mappings . . . . .	33
2.3 Equations with monotone mappings . . . . .	35
2.4 Quasilinear elliptic equations . . . . .	40
2.4.1 Boundary-value problems for 2nd-order equations . . . . .	41
2.4.2 Weak formulation . . . . .	42
2.4.3 Pseudomonotonicity, coercivity, existence of solutions . . .	46
2.4.4 Higher-order equations . . . . .	53
2.5 Weakly continuous mappings, semilinear equations . . . . .	56

2.6	Examples and exercises . . . . .	58
2.6.1	General tools . . . . .	59
2.6.2	Semilinear heat equation of type $-\operatorname{div}(A(x, u)\nabla u) = g$ . . .	62
2.6.3	Quasilinear equations of type $-\operatorname{div}( \nabla u ^{p-2}\nabla u) + c(u, \nabla u) = g$	69
2.7	Excursion to regularity for semilinear equations . . . . .	79
2.8	Bibliographical remarks . . . . .	86
<b>3</b>	<b>Accretive mappings</b>	<b>89</b>
3.1	Abstract theory . . . . .	89
3.2	Applications to boundary-value problems . . . . .	93
3.2.1	Duality mappings in Lebesgue and Sobolev spaces . . . . .	93
3.2.2	Accretivity of monotone quasilinear mappings . . . . .	95
3.2.3	Accretivity of heat equation . . . . .	99
3.2.4	Accretivity of some other boundary-value problems . . . . .	102
3.2.5	Excursion to equations with measures in right-hand sides .	103
3.3	Exercises . . . . .	106
3.4	Bibliographical remarks . . . . .	107
<b>4</b>	<b>Potential problems: smooth case</b>	<b>109</b>
4.1	Abstract theory . . . . .	109
4.2	Application to boundary-value problems . . . . .	114
4.3	Examples and exercises . . . . .	120
4.4	Bibliographical remarks . . . . .	124
<b>5</b>	<b>Nonsmooth problems; variational inequalities</b>	<b>125</b>
5.1	Abstract inclusions with a potential . . . . .	125
5.2	Application to elliptic variational inequalities . . . . .	129
5.3	Some abstract nonpotential inclusions . . . . .	135
5.4	Excursion to quasivariational inequalities . . . . .	144
5.5	Exercises . . . . .	147
5.6	Some applications to free-boundary problems . . . . .	152
5.6.1	Porous media flow: a potential variational inequality . . . .	152
5.6.2	Continuous casting: a nonpotential variational inequality .	156
5.7	Bibliographical remarks . . . . .	159
<b>6</b>	<b>Systems of equations: particular examples</b>	<b>161</b>
6.1	Minimization-type variational method: polyconvex functionals . . .	161
6.2	Buoyancy-driven viscous flow . . . . .	168
6.3	Reaction-diffusion system . . . . .	172
6.4	Thermistor . . . . .	175
6.5	Semiconductors . . . . .	178

<b>II</b>	<b>EVOLUTION PROBLEMS</b>	<b>185</b>
<b>7</b>	<b>Special auxiliary tools</b>	<b>187</b>
7.1	Sobolev-Bochner space $W^{1,p,q}(I; V_1, V_2)$ . . . . .	187
7.2	Gelfand triple, embedding $W^{1,p,p'}(I; V, V^*) \subset C(I; H)$ . . . . .	190
7.3	Aubin-Lions lemma . . . . .	193
<b>8</b>	<b>Evolution by pseudomonotone or weakly continuous mappings</b>	<b>199</b>
8.1	Abstract initial-value problems . . . . .	199
8.2	Rothe method . . . . .	201
8.3	Further estimates . . . . .	215
8.4	Galerkin method . . . . .	221
8.5	Uniqueness and continuous dependence on data . . . . .	228
8.6	Application to quasilinear parabolic equations . . . . .	232
8.7	Application to semilinear parabolic equations . . . . .	239
8.8	Examples and exercises . . . . .	242
8.8.1	General tools . . . . .	242
8.8.2	Parabolic equation of type $\frac{\partial}{\partial t}u - \operatorname{div}( \nabla u ^{p-2}\nabla u) + c(u) = g$ . . . . .	244
8.8.3	Semilinear heat equation $c(u)\frac{\partial}{\partial t}u - \operatorname{div}(\kappa(u)\nabla u) = g$ . . . . .	252
8.8.4	Navier-Stokes equation $\frac{\partial}{\partial t}u + (u \cdot \nabla)u - \Delta u + \nabla \pi = g, \operatorname{div} u = 0$ . . . . .	255
8.8.5	Some more exercises . . . . .	257
8.9	Global monotonicity approach, periodic problems . . . . .	262
8.10	Problems with a convex potential: direct method . . . . .	267
8.11	Bibliographical remarks . . . . .	272
<b>9</b>	<b>Evolution governed by accretive mappings</b>	<b>275</b>
9.1	Strong solutions . . . . .	275
9.2	Integral solutions . . . . .	280
9.3	Excursion to nonlinear semigroups . . . . .	286
9.4	Applications to initial-boundary-value problems . . . . .	291
9.5	Applications to some systems . . . . .	295
9.6	Bibliographical remarks . . . . .	302
<b>10</b>	<b>Evolution governed by certain set-valued mappings</b>	<b>305</b>
10.1	Abstract problems: strong solutions . . . . .	305
10.2	Abstract problems: weak solutions . . . . .	309
10.3	Examples of unilateral parabolic problems . . . . .	313
10.4	Bibliographical remarks . . . . .	318
<b>11</b>	<b>Doubly-nonlinear problems</b>	<b>321</b>
11.1	Inclusions of the type $\partial\Psi(\frac{d}{dt}u) + \partial\Phi(u) \ni f$ . . . . .	321
11.1.1	Potential $\Psi$ valued in $\mathbb{R} \cup \{+\infty\}$ . . . . .	321
11.1.2	Potential $\Phi$ valued in $\mathbb{R} \cup \{+\infty\}$ . . . . .	327
11.1.3	Uniqueness and continuous dependence on data . . . . .	332

11.2	Inclusions of the type $\frac{d}{dt}E(u) + \partial\Phi(u) \ni f$ . . . . .	334
11.2.1	The case $E := \partial\Psi$ . . . . .	335
11.2.2	The case $E$ nonpotential . . . . .	339
11.2.3	Uniqueness . . . . .	341
11.3	2nd-order equations . . . . .	342
11.4	Exercises . . . . .	351
11.5	Bibliographical remarks . . . . .	355
<b>12</b>	<b>Systems of equations: particular examples</b>	<b>357</b>
12.1	Thermo-visco-elasticity . . . . .	357
12.2	Buoyancy-driven viscous flow . . . . .	361
12.3	Predator-prey system . . . . .	365
12.4	Semiconductors . . . . .	368
12.5	Phase-field model . . . . .	372
12.6	Navier-Stokes-Nernst-Planck-Poisson-type system . . . . .	376
	<b>References</b>	<b>383</b>
	<b>Index</b>	<b>399</b>